On Multiplicative Showers

J. F. CARLSON AND J. R. OPPENHEIMER
University of California, Berkeley, California
(Received December 8, 1936)

In I we discuss the status of the quantum theoretic formulae for pair production and radiation in the domain of cosmic-ray energies, and the relevance of these processes to an understanding of showers and bursts. In II we give a qualitative estimate of the course implied by the theory for a shower or burst built up by multiplication from a very energetic primary; we then set up the diffusion equations for the equilibrium of electrons and gamma-rays, and show how these can be simplified. In III we carry through the analytic solution of the diffusion equations, and find the distribution of electrons and gamma-rays as a function of their energy, the primary energy, and the thickness and atomic number of the matter traversed. We treat the effect of ionization losses on the shower, calculate the amount of radiation of low energy to be expected, and treat transition effects in passing from one substance to another. In IV we discuss the results of the calculations, and give a summary of the conclusions to which they lead, and the difficulties.

I

IN nuclear fields, gamma-rays produce pairs, and electrons lose energy by radiation. The formulae which have been deduced[1] from the quantum theory give for the probability of these processes values which, for sufficiently high energies, no longer depend upon the energy of the radiation. Because of this, the secondaries, produced by a photon or electron of very high energy, will be nearly as penetrating as the primary, so that the primary energy will soon be divided over a large number of photons and electrons. It is this development and absorption of showers which we wish to investigate.

The finite limiting cross sections for radiative loss and for pair production essentially limit the penetrating power of electrons and photons; as we shall see, 20 cm of Pb should absorb practically all such radiation if the primary energies are $<10^5$ Mev. From this one can conclude, either that the theoretical estimates of the probability of these processes are inapplicable in the domain of cosmic-ray energies, or that the actual penetration of these rays has to be ascribed to the presence of a component other than electrons and photons. The second alternative is necessarily radical; for cloud chamber and counter experiments show that particles with the same charge as the negative electron belong to the penetrating component of the radiation; and if these are not electrons, they are particles not previously known to physics.[2]

Direct evidence for the approximate validity of the theoretical formulae is provided by the latest studies of Anderson and Neddermeyer[3] on the energy loss and pair production of electrons of energy up to 400 Mev. This evidence is still incomplete; yet it affords absolutely no indication of a breakdown of the theoretical formulae. Since there is good evidence from the altitude and latitude curves of cosmic-ray ionization, as well as from the transition curves for showers and bursts, of a component in the cosmic rays which is strongly absorbed and yet has a very high energy, it seems of interest to investigate in detail the consequences which the theoretical formulae imply for the degradation, multiplication and absorption of such radiation. We shall find in this way a model for the building up and absorption of large showers and bursts which in many important respects agrees with what is found experimentally.[4] From this we should like to derive on the one hand a further argument for the qualitative validity of the theoretical formulae, and on the other for the often repeated suggestion that many showers are built up by a long succession

[1] An account of the results and of the theory upon which they are based may be found in Heitler's book, *The Quantum Theory of Radiation* (Oxford, 1936).

[2] Thus Williams first suggested that penetrating cosmic rays were protons, positive and negative: E. J. Williams, Phys. Rev. **45**, 729 (1934). Cloud chamber evidence would favor a particle of smaller mass.

[3] C. D. Anderson and S. Neddermeyer, Phys. Rev. **50**, 263 (1936).

[4] An account of the experimental findings on showers and bursts and their transition effects may be found in Geiger's article, "Die Sekundaer Effekte der Kosmischen Ultrastrahlung," Naturwiss. (Leipzig, 1935).

of simple elementary processes, and not by the simultaneous ejection of a huge number of particles in one elementary act.

Here a certain caution is necessary. Cloud chamber observations have shown the existence of two fairly well differentiated types of shower.[5] In one of these, and by far the more common, only electrons, positrons and γ-rays appear to take part; the shower particles are, except for those of very low energy, well collimated, with transverse momenta of the order of a few million volts; the showers can often be seen to increase in passing through matter, and typically, if they are large, have no well-defined focus. It is these showers for which our calculations will give us some understanding. In the other and rarer type of shower, transverse momenta of the order of 100 Mev are common; the shower is usually not collimated at all; heavy recoil particles are frequently seen; and the total number of particles is usually small. It is natural to ascribe these showers to the interaction of heavy and light particles, and to accept the arguments which Heisenberg[6] has advanced to show that for such processes the probability of ejection of several particles should be of the same order of magnitude as that of one. It would seem, however, that Heisenberg's attempt to interpret on this basis the larger showers and bursts as highly multiple elementary processes is without cogent experimental foundation; and we believe that in fact it rests on an abusive extension of the formalism of the theory of the electron neutrino field.

One can then believe in the applicability of the following analysis to cosmic-ray photons and electrons and their showers, only if he admits the presence of another component to which the analysis is not at all applicable, and of other types of elementary processes, which essentially involve the heavy particles and their coupling with electrons, and which find no place in this treatment.

II

We are interested then in what happens when an electron (or positron or photon) of very high initial energy passes through matter. We shall consider only the three elementary processes of pair production by photons, radiation by electrons, and ionization losses by electrons, the first two because they dominate the multiplication which makes the shower, the third because it limits the size of the shower and absorbs it. We shall not consider relatively rare multiplicative processes, such as the production by electrons of high energy electronic secondaries, the direct production of pairs by electrons, or the Compton effect. We shall suppose that all high energy particles come off forward, and neglect the angular divergence of the shower. We shall not try to treat in detail radiation of such low energy that these simplifications are invalid.

To define the probabilities of the elementary processes, we shall use simplified formulae which approximate closely to the limiting forms which the theory gives for high energies.[1]

Thus for the probability that an electron (or positron) radiate an energy in the range $E, E+\Delta E$ in passing through a thickness Δx of matter of atomic number Z, nuclear density N, we shall take

$$P\Delta x\Delta E=\frac{K\Delta x\Delta E}{E} \quad \text{with} \quad K=\frac{4Z^2e^6N}{\hbar m^2C^5}\ln\frac{200}{Z^{\frac{1}{3}}}. \tag{1}$$

For the probability that a gamma-ray of energy E make a pair of energy E', $E-E'$, we take

$$P'\Delta x\Delta E'=(K'\Delta x\Delta E')/E;\quad E'<E;\quad K'/K=\sigma\sim\tfrac{2}{3}. \tag{2}$$

It is convenient to measure length in terms of a variable $t=xK$. For Pb the unit of length is $\sim\frac{1}{2}$ cm, for water it is about 0.4 m. For elements as heavy as Pb, the constants K and K' can hardly be regarded as known within 20 percent. Formula (1) is a good approximation as long as the electronic energy is >20 Mev; formula (2) begins to give too large results for $E<50$ Mev, but is off by only a factor $\frac{3}{5}$ at $E=25$ Mev. As for ionization losses, we shall suppose them to be independent of energy, and evaluate them for those energies, where their effect will be important. Thus we shall write for electrons

$$\frac{\partial E}{\partial t}=-\beta,\quad \beta=\frac{4\pi NZe^4}{KmC^2}\ln\frac{\beta}{ZRh}. \tag{3}$$

[5] Anderson and Neddermeyer, see reference 3; Brode, MacPherson and Starr, Phys. Rev. **50**, 581 (1936).

[6] W. Heisenberg, Zeits. f. Physik **101**, 533 (1936).

For Pb, β has the value 6.5 Mev. It is quite closely inversely proportional to Z.

For primary energies which are not too high, one can carry through the calculations step by step, finding how many gamma-rays are produced by the primary, how many of these are absorbed by pair production, how many gamma-rays these in turn produce. This procedure was used in computing the number of pairs theoretically to be expected in the electron traversals studied by Anderson and Neddermeyer; it has been used by Nordheim, and by Heitler and Bhabha,[7] though here with neglect of ionization losses. It clearly becomes prohibitively laborious for thicknesses and energies of the order of those involved in large showers and in bursts; and we shall try instead to solve the diffusion equations implied by (1), (2) and (3), and so obtain an insight into the course of the shower for large thicknesses and high primary energies. An extremely rough indication of what we may expect to find we can see from a quite simple argument.

Both radiation and pair production give two rays for one in the shower. Each process of this kind has about an even chance of happening in a distance $t=1$ (somewhat less than even for pair production, somewhat more for γ-radiation). Thus the order of magnitude of the total number of electrons and γ-rays to be expected at a thickness t is 2^t. The order of magnitude of the energy loss in thickness Δt is thus $\sim \frac{1}{2}\beta 2^t \Delta t$. When the integral of this is equal to the primary energy E_0, the shower will be absorbed. Thus if T is the distance to which the shower penetrates

$$\beta 2^T \sim 2E_0 \ln 2. \tag{4}$$

From this it follows that T will increase logarithmically with E_0 and that it will decrease slowly with decreasing Z; that the number of particles in the shower will increase with E_0 roughly linearly, and will be roughly proportional to Z. For showers of about 30 particles we should expect the maximum in the shower to come at $T\sim -\log_2 30 \sim 5$, or a little over two cm of Pb or about 7 cm of Fe. These are in fact[4] of the order of the distances at which maxima are found for the transition curves for relatively small showers. For bursts of 1000 particles the transition maxima are found[5] at about twice as great thicknesses, as our rough estimate would suggest.

To set up our diffusion equations, let us write $\gamma(t, E)\Delta E$ for the probable number of gamma-rays to be found at a thickness t in the energy range $E, E+\Delta E$ and $\mathcal{P}(t, E)\Delta E$ for the corresponding number of electrons and positrons. Then

$$\frac{\partial \gamma}{\partial t} = -\sigma\gamma + \frac{1}{E}\int_E^{E_0} \mathcal{P}(t, \epsilon)d\epsilon, \tag{5}$$

$$\frac{\partial \mathcal{P}}{\partial t} = 2\sigma \int_E^{E_0} \frac{\gamma(t, \epsilon)}{\epsilon} d\epsilon + \beta \frac{\partial \mathcal{P}}{\partial \epsilon} + \int_{[E]}^{E_0} \frac{\mathcal{P}(t, \epsilon)}{\epsilon - E} d\epsilon - \mathcal{P}(t, E)\int_{[0]}^{E} \frac{d\epsilon}{\epsilon}. \tag{6}$$

The last two terms in (6) can be combined to give the clearly finite result

$$R = \lim_{\delta\to 0}\left\{\int_{E+\delta}^{E_0} \frac{\mathcal{P}(t, \epsilon)}{\epsilon - E} d\epsilon - \mathcal{P}(t, E)\int_\delta^E \frac{d\epsilon}{\epsilon}\right\}$$
$$= -\mathcal{P}(t, E) \ln \frac{E}{E_0 - E} - \int_E^{E_0} \frac{\partial \mathcal{P}}{\partial \epsilon}(t, \epsilon) \ln\left(\frac{\epsilon}{E} - 1\right) d\epsilon \tag{7}$$

and give the change in pair distribution which comes directly from the radiative losses of the charged particles. The term in β gives the corresponding change from ionization losses, and the first term on the right in (6) gives the pair production by γ-rays. In (5) the first term gives the absorption of γ-rays by pair production, and the second term their replenishment by radiation. These equations are to be solved for the boundary conditions

$$\gamma(t=0, E) = 0; \quad \mathcal{P}(t=0, E) = \Delta(E, E_0), \tag{8}$$

where Δ vanishes except when E is in the immediate neighborhood of E_0, and has the in-

[7] We are indebted to Dr. Nordheim for writing to us of his results. See also Heitler and Bhabha, Nature **138**, 401 (1936). We are further indebted to Heitler and Bhabha for sending us a manuscript of the paper in which they have extended these calculations. Their results differ from ours primarily because of their neglect of ionization losses; apart from this the agreement between their values and ours is excellent. We do not agree with their conclusion that these calculations make it possible to ascribe the greater part of sea-level cosmic radiation to degraded electrons and photons of high initial energy.

tegral 1. It will be convenient, however, to take $\Delta(E_0, E_0)=0$, and to choose for Δ, not a delta function, but

$$\Delta(E, E_0)=\lim_{\alpha\to\infty}\frac{\alpha^n}{E}(\ln E_0/E)^{n-1}\left(\frac{E}{E_0}\right)^\alpha\frac{1}{\Gamma(n)},\quad n\geq 4. \quad (9)$$

The reasons for this choice will be apparent as we develop the solution.

From (5) we may write

$$\mathcal{P}(t, E)=e^{-\sigma t}(\partial^2 z)/(\partial t\partial E) \text{ with } \gamma=-e^{-\sigma t}z/E. \quad (10)$$

It is not possible by differentiation to reduce (6) to a differential equation, because of the occurrence of E in the integrand of (7). We have therefore tried to replace the terms (7) by others, which would lead to a readily soluble system of differential equations, and which would still give a good representation of (7). As we shall see, we can obtain a solution in terms of integrals of elementary functions provided we write in place of (7)

$$R\to a\int_E^{E_0}\frac{\mathcal{P}(t, \epsilon)}{\epsilon}d\epsilon+b\mathcal{P}(t, E)+cE\frac{\partial\mathcal{P}}{\partial E}(t, E)+dE^2\frac{\partial^2\mathcal{P}}{\partial E^2}+\cdots \quad (11)$$

if a, b, c, d are constants. Two elementary conditions now limit their choice.

1. The number of particles leaving a given energy range by radiation must be equal to the number entering some other range.

$$\int_0^{E_0}RdE=0.$$

2. The energy lost in Δx must be equal to that appearing in the γ-rays

$$\int_0^{E_0}REdE=\int_0^{E_0}\mathcal{P}(t, E)dE.$$

If these conditions are to hold for any $\mathcal{P}$, then $c=d=0$, etc. From (1) $a+b=0$; and from (2) $a+b/2=1$; thus we must have $a=-b=2$ and

$$R=2\int_E^{E_0}\frac{\mathcal{P}(t, \epsilon)}{\epsilon}d\epsilon-2\mathcal{P}(t, E). \quad (11a)$$

The use of this form for R corresponds to assuming a uniform distribution of energy losses instead of the $1/E$ law given by (1), and renormalizing to give the correct total energy loss. The production of γ-rays is, however, treated correctly in accordance with (1); it is only the effect of these losses on the redistribution of the electrons themselves that is falsified. Since it is difficult to give an *a priori* estimate of the error thus introduced into the solution, we have used the solutions which we have obtained to compare the values of R given by (7) and by (11a). When t is not too small, most of the particles have of course an energy $E\ll E_0$. For this case we found that the two values of R agreed within less than 5 percent. It is of some interest to ask why this should be.

The answer is to be found in the circumstance that for $E\ll E_0$, our energy distribution curves for $\mathcal{P}$ follow quite closely a $1/E^2$ law as Fig. 4 shows, and as cloud chamber observations suggest that they should. For this law

$$\mathcal{P}(t, E)\sim k(t)/E^2$$

and for $E\ll E_0$, both (7) and (11a) agree in giving

$$R=-k(t)/E^2.$$

The approximation of replacing (7) by (11a) is thus a very good approximation in the range where neither t nor $\ln E_0/E$ are too small; and we shall see that it is only here that our results are of direct physical interest, because for small t and $E\sim E_0$, the fluctuations to be expected from the probable behaviors defined by $\mathcal{P}$ and γ are all-important.

Using (11a) for R, and differentiating (6) we get, if we use (10), the differential equation for z

$$\ddot{z}''+(2-\sigma)\dot{z}''+2\dot{z}'/E-2\sigma z/E^2-\beta\dot{z}'''=0 \quad (12)$$

or, with $\sigma=\frac{2}{3}$,

$$\ddot{z}''+2\sigma\dot{z}''+\frac{2}{E}\dot{z}'-\frac{2\sigma z}{E^2}-\beta\dot{z}'''=0. \quad (12')$$

Here primes denote differentiation with respect to E and dots with respect to t. Eq. (12′) is to be solved with the boundary conditions

$$z(0, E)=0; \qquad \dot{z}'(0, E)=\Delta(E, E_0). \quad (13)$$

III

Let us now write

$$\lambda = \ln (E_0/E)$$

and try to solve (12′) by a Laplace transformation:

$$z = 1/(2\pi i)\int_C (dy/y)e^{\lambda y}F_y(t), \tag{14}$$

where C is some suitably chosen contour in the y plane. From (12′) we then find

$$\frac{1}{2\pi i}\int_C \frac{dy}{y}e^{(2+y)\lambda}\left\{y(y+1)\frac{\partial^2 F_y}{\partial t^2}\right.$$
$$\left.+[2\sigma y(y+1)-2y]\frac{\partial F_y}{\partial t}-2\sigma F_y\right\}$$
$$=-\frac{\beta}{2\pi i E_0}\int_C dy(y+1)(y+2)e^{(3+y)\lambda}\frac{\partial F_y}{\partial t}, \tag{15}$$

where β/E_0 is a very small number. If we neglect these terms, then

$$F_y = M_y e^{\mu(y)t} + N_y e^{\nu(y)t}, \tag{16}$$

where μ, ν are the roots of

$$\kappa^2 + 2(\sigma - 1/y+1)\kappa - 2\sigma/y(y+1) = 0,$$
$$\begin{matrix}\mu\\ \nu\end{matrix} = -\sigma + 1/y+1 \pm [(\sigma - 1/y+1)^2 + 2\sigma/y(y+1)]^{\frac{1}{2}} \tag{17}$$

and for $R(y) > 0$, $\mu > \nu$. The constants M_y, N_y, and the contour C must be chosen to satisfy (13).

It is clear that the integrand of (14) will have branch points in the negative half y plane, and that it will $\rightarrow 0$ as $|y| \rightarrow \infty$. We therefore choose for our contour C a closed path, consisting of a straight line $\delta - i\infty$ to $\delta + i\infty$, and the infinite semicircle from $\delta + i\infty$ through the negative half plane to $\delta - i\infty$. From (13) we get

$$\int_C dy/y e^{\lambda y} F_y(0) = 0,$$
$$\int_C dy e^{\lambda y}\partial F_y/\partial t = -2\pi i E\Delta(E, E_0). \tag{18}$$

If we multiply (18) by $e^{-y_0\lambda}$, with $R(y_0) > \delta$, and integrate over λ from 0 to ∞, we get

$$\int_C \frac{dy}{y-y_0}\frac{F_y(0)}{y} = 0, \quad \frac{1}{2\pi i}\int_C \frac{dy}{y-y_0}\frac{\partial F_y(0)}{\partial t} = -\phi(y_0)$$

$$\text{with}\quad \phi = \lim_{\alpha\rightarrow\infty}\int_0^\infty e^{-y_0\lambda}\frac{\alpha^n}{\Gamma(n)}\lambda^{n-1}e^{-\alpha\lambda}d\lambda$$
$$= \lim_{\alpha\rightarrow\infty}(1+y_0/\alpha)^{-n}. \tag{19}$$

Since F_y and $\partial F_y/\partial t \rightarrow 0$ as $|y| \rightarrow \infty$, and have no singularities in the right-hand half-plane, this gives

$$F_y(0) = 0; \quad \partial F_y(0)/\partial t = -\phi(y). \tag{20}$$

These are the boundary conditions for F. Any other function Δ which gives an $F_y \rightarrow 0$ and $\partial F_y/\partial t \rightarrow 0$ as $|y| \rightarrow \infty$, leaves them without singularities in the half-plane $R(y) > 0$, and whose limit gives $\Delta(E, E_0) = 0$ for $E \neq E_0$, could clearly be used in place of (9). But for the treatment of ionization losses it is convenient to have the first 3 derivatives of Δ vanish as $E \rightarrow E_0$.

Let us now carry through the solution for $\beta = 0$. For this solution we write z_0, γ_0, $\mathcal{P}_0$. From (20)

$$M_y = -\phi/(\mu-\nu), \quad N_y = \phi/(\mu-\nu), \tag{21}$$

$$z_0 = -\frac{1}{2\pi i}\int_C \frac{dy}{y}e^{\lambda y}\frac{e^{\mu t}-e^{\nu t}}{\mu-\nu}\phi. \tag{22}$$

From this we can readily calculate the total energy of gamma-rays and electrons:

$$\int_0^{E_0} EdE[\mathcal{P}_0(t, E) + \gamma_0(t, E)]$$
$$= E_0 e^{-\sigma t}\frac{1}{2\pi i}\int_C \frac{dy}{y}\frac{\phi}{1-y}\frac{(\mu y+1)e^{\mu t}-(\nu y+1)e^{\nu t}}{\mu-\nu}. \tag{23}$$

The integral on the right gives the residue of the integrand at $y = 1$, which is just, as $\alpha \rightarrow \infty$,

$$2\pi i e^{\sigma t} \quad \text{since} \quad \mu = \sigma, \quad \nu = -1 \quad \text{when} \quad y = 1.$$

Thus the total energy remains constant, as it should, when ionization losses are neglected. For this the condition on the constants in (11) $a + b/2 = 1$, is essential.

When t or λ are large, then (22) may most conveniently be evaluated by the saddle point method. The approximation here involved is one which corresponds closely to the limitations imposed upon the physical interpretation of our solution by the fluctuations which must be expected in the actual behavior of the radiation from the average behavior given by our diffusion equations. The effect of these fluctuations can be simply formulated by the physically obvious assertion that the addition or subtraction of a thickness of matter corresponding to $t \sim 1$ has an even chance of not altering the actual distribution of pairs and gamma-rays. The asymptotic form of $\mathcal{P}$, γ for large t thus gives us results whose accuracy corresponds to the applicability of the diffusion equations themselves.

The unique saddle point of the first term of the integrand of (22) lies on the positive real axis, at a point which moves out to larger y values monotonically as t/λ is increased. If we use a subscript s to indicate that functions are to be evaluated at this saddle point, then we find from this term

$$\begin{aligned} z_0 &= -(2\pi\mu_s''t)^{-\frac{1}{2}}\frac{\exp(\lambda y_s+t\mu_s)}{y_s(\mu_s-\nu_s)}, \\ \gamma_0 &= -z_0 e^{-\sigma t+\lambda}/E_0, \\ \mathcal{P}_0 &= \mu_s y_s \gamma_0. \end{aligned} \tag{24}$$

Here $\mu'' = \partial^2\mu/\partial y^2$.

From the second term of (22)

$$\frac{1}{2\pi i}\int_C \frac{dy}{y}\phi\frac{e^{\lambda y+\nu t}}{\mu-\nu}$$

we obtain a contribution which remains less than $Qe^{-\frac{2}{3}t}$, where Q is independent of λ and t, as $t \rightarrow \infty$. For large t this term is thus quite negligible compared to (24).

For $\lambda \gg t$, and $\lambda \ll t$, these expressions (24) can be evaluated analytically. Thus (writing $\sigma = 2/3$).

For $\lambda \gg t$

$$\begin{aligned} z_0 &= 2^{-2}\pi^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}\exp\left[\tfrac{1}{3}t+3^{\frac{2}{3}}\lambda^{\frac{1}{3}}t^{\frac{2}{3}}\right], \\ \mu_s y_s &= 2\cdot 3^{-\frac{2}{3}}\lambda^{-\frac{1}{3}}t^{\frac{1}{3}}, \end{aligned} \tag{25}$$

and for $\lambda \ll t$,

$$\begin{aligned} z_0 &= 2^{-8/3}\pi^{-\frac{1}{2}}3^{\frac{1}{2}}t^{-\frac{1}{6}}\lambda^{-\frac{1}{3}}\exp\left[3\cdot 2^{-\frac{2}{3}}\lambda^{\frac{2}{3}}t^{\frac{1}{3}}\right], \\ \mu_s y_s &= 2^{-\frac{1}{3}}\lambda^{\frac{1}{3}}t^{-\frac{1}{3}}. \end{aligned} \tag{26}$$

For intermediate values we give a plot of y_s against λ/t, from which (24) may be evaluated. It should be observed that $\mu_s y_s$ gives directly the ratio of $\mathcal{P}_0$ to γ_0, and that it never exceeds 2/3. At any energy and thickness $t>1$, there are always more γ-rays than electrons.

The solution we have given can be readily extended to diffusion equations with R of the general form (11); only the dependence of μ and ν on y becomes more complicated.

For the treatment of the ionization losses given by the terms in β, it is simplest to return to (14), and write instead

$$z = \frac{1}{2\pi i}\int_C \frac{dy}{y}e^{\lambda y}\left[F_y^{(0)}(t)+\frac{\beta}{E}F_y^{(1)}(t) + \frac{\beta^2}{E^2}F_y^{(2)}(t)\cdots\right] \tag{27}$$

and leave the contour C unaltered. If for $F_y^{(0)}(t)$ we take (16), (21), then (12′) gives, for the first order terms in β.

$$\frac{\partial^2 F_y^{(1)}}{\partial t^2}+2\left(\sigma-\frac{1}{y+2}\right)\frac{\partial F_y}{\partial t}-\frac{2\sigma}{(y+1)(y+2)}F_y^{(1)} = \frac{1}{2\pi i}\frac{y}{(y+1)(y+2)}\int_C \frac{dy'(y'+1)(y'+2)}{y'-y} \times\frac{\mu(y')e^{\mu(y')t}-\nu(y')e^{\nu(y')t}}{\mu(y')-\nu(y')}\phi. \tag{28}$$

The general solution of this is

$$F_y^{(1)}(t)/y = A_y e^{\mu(y)t}+B_y e^{\nu(y)t}+C_y e^{\mu(y+1)t} + D_y e^{\nu(y+1)t}, \tag{29}$$

where

$$\begin{aligned} A_y &= \phi\frac{\mu(\sigma+\mu y)(y+1)(y+2)}{2(2\sigma+\mu y)(\mu-\nu)}, \\ B_y &= -\phi\frac{\nu(\sigma+\nu y)(y+1)(y+2)}{2(2\sigma+\nu y)(\mu-\nu)}, \end{aligned} \tag{29a}$$

and where C_y and D_y must be chosen to maintain the boundary conditions (13), which require

$$\begin{aligned} C_y &= \frac{A_y\mu(y)+B_y\nu(y)-\nu(y+1)(A_y+B_y)}{\nu(y+1)-\mu(y+1)}, \\ D_y &= \frac{A_y\mu(y)+B_y\nu(y)-\mu(y+1)(A_y+B_y)}{\mu(y+1)-\nu(y+1)}. \end{aligned} \tag{29b}$$

For $R(y) > -1$, C_y and D_y are analytic, except for a simple pole of C_y at

$$y_p = (3-\sqrt{73})/8 \sim -0.69$$

and a simple pole of D_y at

$$y_p' = (3+\sqrt{73})/8 \sim 1.44.$$

The terms of order β in (27) may then be written

$$\frac{\beta}{E_0 2\pi i}\int_C dy e^{(1+y)\lambda}[A_y e^{\mu(y)t} + B_y e^{\nu(y)t} + C_y e^{\mu(y+1)t} + D_y e^{\nu(y+1)t}]. \quad (30)$$

For large λ and t only the term

$$\frac{\beta}{E_0 2\pi i}\int_C dy e^{(1+y)\lambda} A_y e^{\mu(y)t} = \frac{\beta}{E 2\pi i}\int_C dy A_y e^{\lambda y + \mu t} \quad (31)$$

is important.

For the second and fourth terms in (30) we can again show that they remain less than

$$\frac{\beta}{E_0} Q e^{\lambda - 2/3t} \quad \text{as } t \to \infty,\ \lambda \to \infty$$

and are therefore negligible.

The term

$$\Gamma = \frac{\beta}{E_0 2\pi i}\int_C dy C_y e^{(1+y)\lambda + \mu(1+y)t}$$

may be evaluated by deforming the contour C to pass through the saddle point at

$$y_\sigma = -1 + y_s.$$

If we call the path taken along a line parallel to the imaginary axis through y_σ, Σ then

$$\Gamma = \frac{\beta}{2\pi i E_0} P\int_\Sigma dy C_y e^{(1+y)\lambda + \mu(1+y)t} + \frac{\beta}{E_0}\vartheta(y_\sigma - y_p) G_p e^{(1+y_p)\lambda + \mu(1+y_p)t}$$

where
$$\vartheta(x) = \begin{cases} 0 & x > 0 \\ \tfrac{1}{2} & x = 0 \\ 1 & x < 0 \end{cases}$$

and
$$G_p = \lim_{\epsilon \to 0}(\epsilon C_{y_p + \epsilon}).$$

Each of the terms in Γ gives a contribution negligible for $\lambda \to \infty$ compared to (31). From the saddle point integral we get a contribution

$$< Q(\lambda, t)\frac{\beta}{E_0} e^{y_s \lambda + \mu_s t},$$

where Q is an algebraic function of λ and t; from the residue we get

$$< Q_1 \frac{\beta}{E_0} e^{y_s \lambda + \mu_s t + (1+y_p)\lambda}$$

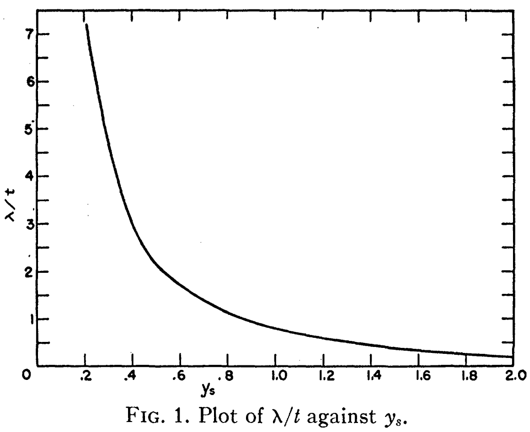

FIG. 1. Plot of λ/t against y_s.

with Q_1 a constant. Both of these are small compared to the term

$$\frac{\beta}{E_0}(2\pi\mu_s'' t)^{-1/2} A_{y_s} e^{(1+y_s)\lambda + \mu_s t},$$

which we get from the saddle point integral (31).

From (31), and the application of the saddle point method to the integral occurring in it, we can readily see by what factors the values of z_0, γ_0, $\mathcal{P}_0$ given in (24) must be modified:

$$z = z_0(1 - \beta\tau_s/E)\,; \quad \gamma = \gamma_0(1 - \beta\tau_s/E)\,;$$
$$\mathcal{P} = \mathcal{P}_0(1 - \beta(1 + 1/y_s)\tau_s/E)\,;$$
$$\mathcal{P} = \gamma\mu_s y_s(1 - \beta\tau_s/y_s E), \quad (32)$$

where
$$\tau = \frac{y(y+1)(y+2)(\sigma + \mu y)}{2(2\sigma + \mu y)}.$$

From (24) and (32) we can thus give $\mathcal{P}$, γ as functions of t and E. The ionization correction still further increases, of course, the proportion of γ-rays in the radiation.

From (32) we see that our approximate calculation of the effect of ionization losses is based upon the smallness of β/E. This parameter is about 12 percent for Pb, 35 percent for Fe, at $E = 50$ Mev. Clearly the treatment here given is limited to energies $E > \beta$, but within this limit it can tell us what the effect of these losses will be.

In fact (22) and (31) give the first two terms in the asymptotic expansion in β/E of a solution of (12′):

$$z = -\frac{1}{4\pi^2}\int_C \frac{dy}{y}\frac{e^{\mu t + \lambda y}}{\mu - \nu}\phi\Gamma(-y)\int_S ds(-s)^y e^{-s} \times {}_2F_1\left(y, y+2; y+1+g; -\frac{\beta\mu y g s e^{\lambda}}{2\sigma E_0}\right). \quad (33)$$

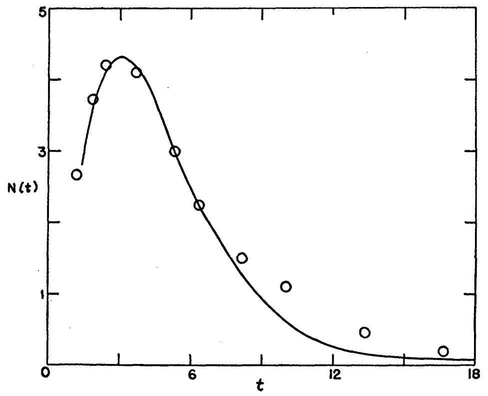

FIG. 2. Plot of total number of electrons $N(t)$ against t, for air, $E_0=2500$ Mev, computed from (36). The circles are from the experimental results of Pfotzer,* on the variation of vertical coincidence counting rate in the upper atmosphere, at magnetic latitude $\sim 50°$N. At this latitude the earth's field will just admit electrons of energy 2500 Mev. The deviations from the curve for $t>8$ may indicate the presence of some electrons of higher energy, as well as a penetrating component.

Here the contour S in the s plane is a simple loop from $+\infty$ counterclockwise around the origin and back to $+\infty$; ${}_2F_1$ is the hypergeometric series and

$$g=\sigma(1+y)/(\sigma+\mu y).$$

That (33) is a solution of (12′) may be verified by direct substitution. It is not of course the solution satisfying the boundary conditions (13); but an iteration of the arguments just given in connection with (31) indicates that the terms which must be added to satisfy (13) give for $t>1$, $E\ll E_0$, a negligible contribution.

For heavy elements such as Pb, (31) gives an adequate approximation to (33) for all energies high enough to make the use of (2) permissible: the pair production formulae break down before the ionization correction becomes very large. For light elements, for the atmosphere, we may apply (33) for $E<\beta$. To do this we use the analytic continuation of the hypergeometric series, and evaluate the integral over y by the saddle point method. For $E/\beta<1$ we thus find

$$\gamma(E,t)=e^{\lambda_\beta y_s+(\mu_s-\sigma)t}\frac{(2\pi\mu_s''t)^{-\frac{1}{2}}}{y_s(\mu_s-\nu_s)}\times\frac{\Gamma(1+y_s+g_s)}{\Gamma(1+y_s)\Gamma(2+y_s)\Gamma(1+g_s)}\frac{1}{E},$$
$$\mathcal{P}(E,t)=\frac{2\sigma E}{\beta}\gamma(E,t)\ln\frac{\beta}{E}. \tag{34}$$

* Zeits. f. Physik **102**, 41 (1936), Fig. 1, p. 42.

Here $\lambda_\beta=\ln(2\sigma E_0)/(\beta\mu_s y_s g_s)$; and the saddle point y_s is given as the real positive root of

$$\frac{d}{dy}\left\{y\ln\frac{2\sigma E_0}{\beta\mu y g}+\mu t\right\}=0 \tag{35}$$

and may be found from Fig. 1.

The energy dependence of γ and $\mathcal{P}$, which for $E>\beta$ is roughly given by k/E^2, becomes, if $E<\beta$, k_1/E, $k_2\ln\beta/E$, respectively. The increase in γ and $\mathcal{P}$ with decreasing energy will be still further reduced at still lower energies by the absorption of the γ-rays by Compton effect. The total number of electrons in the range $0-\beta$ depends upon β essentially as

$$e^{\lambda_\beta y_s}$$

and its maximum as a function of t is quite closely inversely proportional to β or proportional to Z.

The total number of electrons is easy to compute from (33):

$$N(t)=\int_0^{E_0}\mathcal{P}(E,t)dE=-e^{-\sigma t}\dot{z}(E=0,t)$$
$$=\frac{\mu_s(2\pi\mu_s''t)^{-\frac{1}{2}}}{y_s(\mu_s-\nu_s)}\frac{\Gamma(1+y_s+g_s)}{\Gamma(1+y_s)\Gamma(2+y_s)\Gamma(1+g_s)}\times e^{\lambda_\beta y_s+(\mu_s-\sigma)t}, \tag{36}$$

where y_s is given again by (35). The maximum value of $N(t)$ occurs for values of t slightly smaller than λ_β. We give Fig. 2 a plot of $N(t)$ against t, for water, for which we have taken $\beta=90$ Mev, and t is measured in units of $\sim 0.4m$. The primary energy is $\sim 2.5\times 10^3$ Mev. The maximum of N occurs at about 1.2 meters. It would seem probable that the latitude sensitive transition effects of the cosmic-ray ionization in the upper atmosphere are to be interpreted on this basis.

It is futile to apply (33) to elements as heavy as Pb, because for energies of the order of β the radiative formulae (1), (2) and the diffusion equation based on them become quite wrong. As is well known, the actual behavior of radiation in the range 1–25 Mev in Pb is extremely complicated. Since the absorption coefficient for γ-rays has a minimum value which for Pb lies at about 3 Mev, we know that for sufficiently great thicknesses of matter γ-radiation of roughly this energy will predominate over γ-rays of higher or

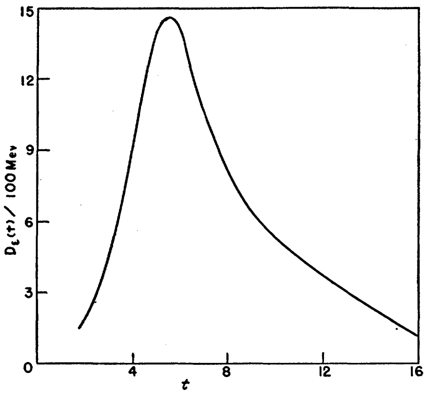

FIG. 3. A plot of the energy degraded into radiation of energy <25 Mev per unit t, against t. The plot is for Pb, $E_0=2\times10^4$ Mev. The abscissae are in units of $\frac{1}{2}$ cm, the ordinates in units 100 Mev. The number of low energy electrons to be expected at t is roughly 0.15 $D_{25}(t-3)$.

lower energy and over electrons of all energies. A rough estimate of the number of low energy electrons we can get by computing the total energy degraded into the low energy region per unit thickness. Let us call the energy so degraded into an energy region $0<E<\epsilon$, $D_\epsilon(t)$. This energy has ultimately to be absorbed by the ionization losses of electrons. The number of electrons which in a distance $t=1$ would lose by ionization an energy $D_\epsilon(t)$ is $1/\beta D_\epsilon(t)$; it must be remembered, however, that because of the relatively great penetration of the low energy gamma-rays (amounting for Pb to $t=3$–4), the low energy electrons actually present in the radiation at some fixed thickness $t=T$ must be evaluated, not from $D_\epsilon(T)/\beta$, but from some appropriately retarded value $D_\epsilon(T-r)/\beta$, where r gives the effective mean penetration of the low energy radiation, and may be of the order of 2–3 for Pb. It is in this sense that the curves we give for $D_\epsilon(t)$ are to be interpreted. For the interpretation of observations made with the high pressure, thick-walled ionization chambers which are used in the study of bursts, the variation of $D_\epsilon(t)$ with t should itself give a valid estimate of the probable variation of the recorded size of the burst with absorber thickness.

The calculation of $D_\epsilon(t)$ is straightforward. The total energy lost by the radiation of energy $>\epsilon$ per unit t is

$$D^{(1)}(t)=-\frac{d}{dt}\int_\epsilon^{E_0}[\mathcal{P}(E,t)+\gamma(E,t)]EdE. \quad (37)$$

The part of this which is accounted for by ionization losses is

$$D_\epsilon^{(2)}(t)=\int_\epsilon^{E_0}\beta\mathcal{P}(E,t)dE. \quad (38)$$

Then
$$D_\epsilon(t)=D_\epsilon(t)^{(1)}-D_\epsilon(t)^{(2)}. \quad (39)$$

In this way we find, from (39), (32) and (24)

$$D_\epsilon(t)=-e^{-\sigma t}z_0\left\{\frac{(\mu_s-\sigma)(1+\mu_s y_s)}{1-y_s}\epsilon - [(\sigma-\mu_s)[1+(1+y_s)\mu_s]y_s^{-1}\tau+\mu_s]\beta\right\}. \quad (40)$$

Here z_0 is given by (24), and is to be evaluated, as are μ_s, y_s and τ, for $E=\epsilon$, $\lambda=\ln E_0/\epsilon$, $t=t$. It will be observed that the effect of the terms in β in (40) is to increase $D_\epsilon(t)$ when $\lambda\gg t$, and to decrease it when $\lambda\ll t$. We give in Fig. 3 a plot of $D_\epsilon(t)$ for Pb, $\epsilon=25$ Mev, $E_0=2\times10^4$ Mev.

Let us now restrict ourselves to $E>\beta$, and ask how the shower will be altered by changes in the initial radiation.

A. If the incident radiation is a γ-ray of energy E_0 and not an electron, the boundary conditions (20) must be altered to

$$F_y(0)=\phi y; \qquad \partial F_y(0)/\partial t=0. \quad (41)$$

This gives us

$$z_{0[\gamma]}=\frac{1}{2\pi i}\int_C\frac{\nu e^{\mu t}-\mu e^{\nu t}}{\mu-\nu}dy e^{\lambda y}\phi, \quad (42)$$

from which it follows that for $t>1$, the number of gamma-rays and electrons will be changed by a factor $-\nu_s y_s$ from the values given by (32). This factor varies from 0.6 to 1.6 over the range of values of t/λ for which the shower is appreciable. The course and magnitude of the shower will thus depend very little on whether it is started by an electron or a gamma-ray. The maximum of the shower will come for very slightly higher t values when it is initiated by a γ-ray.

B. If the incident radiation is a group of m electrons of energy E_0/m, then from (24)

$$\left.\begin{aligned}\mathcal{P}_{0[m]}&=m^2\mathcal{P}_0(\lambda-\ln m,t);\\ \gamma_{0[m]}&=m^2\gamma_0(\lambda-\ln m,t)\end{aligned}\right\} \quad (43)$$

and insofar as $\mathcal{P}(\lambda, t)$ is quite closely of the form $k(t)e^{-2\lambda}$, for small variations of λ,

$$\mathcal{P}_{0[m]}=\mathcal{P}_0$$

for small m, and the shower will be substantially the same as if one electron of energy E_0 had started it.

C. As long as we neglect the ionization terms, we can get no "transition effects" when a shower, built up in one substance, passes into another. But the terms in β do give us such effects, in apparent qualitative agreement with what is found experimentally. Let the radiation go a distance t_1 (measured in units approximate to the first substance) in (1), and then a distance t_2 (in units for (2)) in (2). Then the terms independent of β in z are, for $t_1+t_2>1$.

$$z_0=-\frac{1}{2\pi i}\int_C \frac{dy}{y}\phi\frac{e^{\lambda y+\mu(t_1+t_2)}}{\mu-\nu}. \tag{44}$$

The boundary conditions

$$\mathcal{P}(E, t_1) \quad \text{in} \quad (1)=\mathcal{P}(E, t_1) \quad \text{in} \quad (2);$$
$$\gamma(E, t_1) \quad \text{in} \quad (1)=\gamma(E, t_1) \quad \text{in} \quad (2)$$

give then, for $t_1>1$, for the terms in z proportional to β:

$$\frac{1}{2\pi i}\int_C dy\phi e^{\lambda(y+1)}\bigg\{\beta_2 A_y e^{\mu(y)(t_1+t_2)} +(\beta_1-\beta_2)A_y e^{\mu(y)t_1}\bigg[\frac{\mu(y)-\nu(y+1)}{\mu(y+1)-\nu(y+1)}e^{\mu(y+1)t_2} -\frac{\mu(y)-\mu(y+1)}{\mu(y+1)-\nu(y+1)}e^{\nu(y+1)t_2}\bigg]\bigg\}. \tag{45}$$

which for $t_2=0$ is just the ionization correction appropriate to (1), and which for $t_2\rightarrow\infty$ approaches that appropriate to (2). If $Z_1>Z_2$; $\beta_1<\beta_2$ and $\mathcal{P}$ is decreased by the transition; if $Z_2>Z_1$, $\mathcal{P}$ is increased. The transient terms which give the transition fall off, relative to the main terms, as

$$e^{[\mu(y_s+1)-\mu(y_s)]t_2} \quad \text{and} \quad e^{[\nu(y_s+1)-\nu(y_s)]t_2}.$$

Thus the second term will be damped out for $t_2\sim\frac{1}{2}$, but the first term will persist until $t_2\sim 1.5-2$, the transition thickness is therefore of this order of magnitude. For $E_0=2\times10^4$ Mev, $t_1=6$ the transition from Fe to Pb should increase the number of electrons with energy>50 Mev by about 35 percent.

It must be remembered that the actual transition effects observed are complicated by the fact that the high energy radiation which produces the showers is to some extent being regenerated in the material in which the showers are formed.

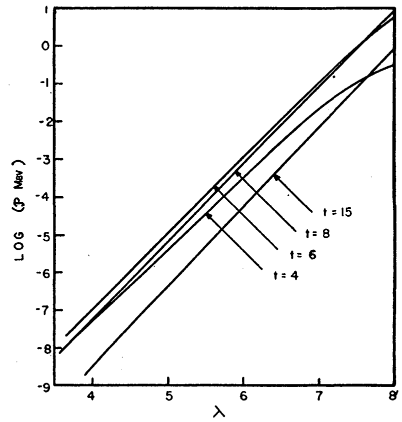

FIG. 4. A logarithmic plot of energy distributions for shower electrons, made for Pb, $E_0=1.5\times10^5$ Mev and $t=15$, $t=8$, $t=6$, and $t=4$. Abscissae are $\lambda=\ln E_0/E$; ordinates are natural logarithms of $\mathcal{P}$ measured in (Mev)$^{-1}$. A straight line of slope 2 would correspond to a $1/E^2$ distribution law.

IV

In Fig. 4 we give typical energy distribution curves for the pairs, for Pb, and a primary energy of 1.5×10^5 Mev for $t=15$, 8, 6, 4. In the neighborhood of $\lambda=8$, the ionization corrections are appreciable, but for $\lambda<6$ the curves will hold for all substances. It will be observed that these curves follow quite closely what we would get from a distribution law k/E^2; thus except for $t=4$ they may all be represented by a law of this form in which the exponent of E never differs from -2 by more than five percent. In fact this law gives a very good approximation to the energy distribution for $t>.7\lambda$, and for all energies $E\ll E_0$ but $\gg\beta$. We have already pointed out that it is for this reason that the simplification of the diffusion equations made in replacing (7) by (11) is permissible. Cloud chamber studies[8] of the energy distribution of shower electrons seem to fit this law quite well for $E>25$ Mev, but the experi-

[8] C. D. Anderson and S. Neddermeyer, Int. Conf. Physics, London 171 (1934).

mental conditions here are clearly not such as to make this agreement very significant.

Because of the form of the distribution law, a plot of $\mathcal{P}E$ will give a good estimate of the total number of electrons to be expected with energies $\geq E$. For Pb, $E=50$ Mev, and several values of E_0 varying from 2700 Mev up to 1.1×10^6 Mev we give such plots against t (Fig. 5). The rapid increase with E_0 of the total number of particles, and the slow shift in the position of the maximum, confirm the qualitative arguments of §2. We would like to interpret in terms of this shift the well-known observation that the optimum transition thickness for bursts is considerably larger than for showers. In fact, the agreement between the experimental values of the optimal transition thicknesses (~ 2 cm Pb for showers, ~ 5 cm Pb for bursts) with the position of the maxima of these curves, seems to us a strong argument for the correctness of the model we are treating, and of the validity of the high energy formulae we have used. In particular the experiments of Nie,[9] in which it is shown that a burst generated in a suitable layer of matter may sometimes have its magnitude very much increased by the interposition of a few cm of Pb find a very natural interpretation in terms of these theoretical curves. It must be remembered, however, that the experimental transition curves, which give the dependence on t of the probability of finding a shower or burst whose magnitude exceeds a lower limit defined by the experimental arrangement are not strictly comparable with the curves of Fig. 5 which give the variation, for fixed E_0 of the probable number of electrons.[10] For in making this comparison the initial distribution of the radiation over E_0 must clearly be taken into account. We want here, too, to point out that the actual behavior of the radiation will fluctuate about that given by our curves, and that the order of magnitude of the probable fluctuations can be estimated by shifting the curves by $\Delta t \sim \pm 1$.

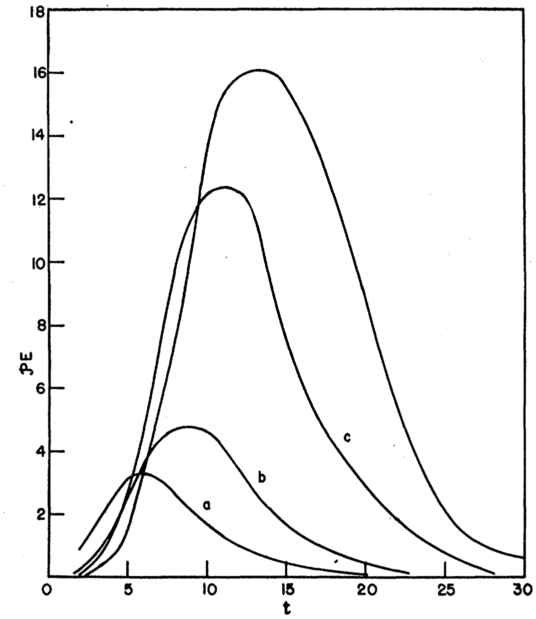

FIG. 5. Plots against t of $\mathcal{P}\cdot E$ for $E=50$ Mev in Pb. (a) $E_0=2.7\times 10^3$, (b) $E_0=2\times 10^4$, (c) $E_0=1.5\times 10^5$; (d) $E_0=1.1\times 10^6$. Abscissae are in units of $\frac{1}{2}$ cm; ordinates in units of 1, 4, 10, and 50 for (a), (b), (c) and (d), respectively. These plots also give the number of electrons of energy $\geqslant 50$ Mev to be expected as a function of t.

The application of our methods to E_0 as low as 2700 Mev may seem unjustified. Here our results, however, agree reasonably with those computed by Bhabha and Heitler[11] with neglect of ionization losses and of processes of high order. For 50 Mev the inclusion of ionization losses reduces the number of electrons to be expected by 15–20 percent.

It may be helpful to give a brief summary of the general results. For any absorber we measure length (t) in units which are proportional, roughly, to $Z^2\rho/A$, where Z is the nuclear charge, A the atomic weight, and ρ the density, and we define the characteristic energy β (see (3)), which varies about like $1/Z$. Then,

(1) The number of electrons per unit energy is about inversely proportional to the square of the energy, as long as $\beta < E \ll E_0$, $t > \frac{1}{2}\lambda$.

(2) For given energy and $t>1$, there are always more γ-rays than electrons; where the

---

[9] Nie, Zeits. f. Physik 99, 776 (1936).

[10] When counters are used to detect showers, the angular divergence of the shower rays is necessarily exploited. If the theory here developed is at all correct, most of these rays must have a relatively low energy, of the order of several million volts. In fact only cloud chamber observation can tell us much about radiation of much higher energy.

[11] Reference 8. Heitler and Bhabha give for $E_0=2.7\times 10^3$ Mev, a value $\mathcal{P}E=4$, maximum for $t=4$; we get $\mathcal{P}E=4.5$ at $t=4$. This discrepancy, and the fact that we find the maximum at $t\sim 5$, are both in the direction to be expected from the effects of higher order processes.

shower is near its maximum, this ratio varies from 1.5–2.

(3) For $E \gg \beta$, the distribution curves plotted against t are the same for all absorbers.

(4) The number of particles of energy greater than $E_1 > \beta$ passes through a maximum for a value of t which increases logarithmically with E_0/E_1 and is always quite close to $\ln_2 E_0/E_1$.

(5) The maximum number of particles with an energy less than some small multiple of β is attained for values of t which are slightly smaller than $\ln E_0/\beta$, and decrease slowly with decreasing Z. The total number of particles of energy in this range is about inversely proportional to β, or proportional to Z.

(6) The maximum size of the shower is limited only by E_0 with which it increases not quite linearly: thus an increase in E_0 by a factor of 100 gives an increase in shower size of about 70.

(7) If the initial energy E_0 is in an incident γ-ray, or is divided among a few electrons and gamma-rays, the course of the shower will be essentially unaltered.

(8) Passage of a shower from one material (1) to another (2) will increase the size of the shower if $Z_2 > Z_1$, decreases it if $Z_1 > Z_2$. The transition takes place in a thickness $t_2 \sim 1\frac{1}{2}$. All of these results apply only for energies E_0 above 10^3 Mev.

In this paper we have altogether neglected the question of how such high energy electrons and γ-rays can get down through the atmosphere. How serious this difficulty is we can see from (32), which tells us that for every electron of energy $\sim 2 \times 10^5$ Mev which hits the earth vertically, only 0.15 electron of energy >550 Mev will survive at the earth's surface. This difficulty is made even sharper when we consider the form of the shower curves for great thicknesses $t > 30$, or the showers and bursts reported under very great thicknesses of absorber. In fact, although when we go up far in the atmosphere, the showers, and still more markedly, the bursts, increase more rapidly than the total cosmic-ray ionization, below the atmosphere they do not fall off much more rapidly than this ionization. This suggests that, in addition to primary electrons and perhaps γ-rays, which are able to produce multiplicative showers directly, there is another cosmic-ray component, slowly absorbed, which is responsible for the continuation of the showers under thicknesses of absorber to which no electron or photon can itself penetrate. Some suggestions which we think relevant to the solution of this problem will be discussed in another paper.